Sidon spaces and Cyclic Subspace Codes

Chiara Castello

Joint work with Olga Polverino, Paolo Santonastaso and Ferdinando Zullo

Università degli Studi della Campania “Luigi Vanvitelli”

Finite geometry and friends
A Brussels summer school on finite geometry
Vrije Universiteit Brussel

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$\mathcal{C} \subset \mathcal{G}_q(n, m)$, $(\mathcal{C}, d)$ *Constant dimension subspace code*

$$d(\mathcal{C}) := \min\{d(U, V) \,:\, U, V \in \mathcal{C}, U \neq V\}$$

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parameters of $\mathcal{C}$ $[n, d(\mathcal{C}), |\mathcal{C}|, m]_q$

Cyclic subspace codes

$\mathcal{C} \subseteq \mathcal{G}_q(n, m)$ is cyclic if

$$\forall \alpha \in \mathbb{F}_{q^n}^*, \forall V \in \mathcal{C} \quad \Longrightarrow \quad \alpha V \in \mathcal{C}$$

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$$\mathcal{C} = Orb(V) \quad [n, 2m - 2M, \tfrac{q^n-1}{q^s-1}, k]_q$$

$$M := \max\{\dim(\alpha V \cap \beta V) \ : \ \alpha, \beta \in \mathbb{F}_{q^n}^*, \alpha V \neq \beta V\}$$
$$s := \max\{t \mid n \ : \ V \quad \text{is} \quad \mathbb{F}_{q^t} - linear\}$$

One-Orbit Cyclic Subspace Codes

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Next Case $\longrightarrow$ M=1 (and s=1)

Problem: To look for
$\mathcal{C}$ one-orbit cyclic subspace code $\longrightarrow [n, 2m-2, \frac{q^n-1}{q-1}, m]_q$

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Conjecture (A.-L. Trautmann, F. Manganiello, M. Braun and J. Rosenthal)

For any prime power q and positive integers m, there exists a cyclic subspace code $\mathcal{C} \subset \mathcal{G}_q(n, m)$ of minimum distance $2m-2$ and size $\frac{q^n-1}{q-1}$.

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Property (C. Bachoc, O. Serra, G. Zémor)

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for every $a, b, c, d \in V$,

$$ab = cd \quad \Longrightarrow \quad \{a\mathbb{F}_q, b\mathbb{F}_q\} = \{c\mathbb{F}_q, d\mathbb{F}_q\}$$

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$ab = cd \quad \Longrightarrow \quad \{a\mathbb{F}_q, b\mathbb{F}_q\} = \{c\mathbb{F}_q, d\mathbb{F}_q\} \Rightarrow$ V is a *Sidon space*

Sidon spaces and optimal one-orbit subspace codes

Property (R.M. Roth, N. Raviv, I. Tamo 2017)

$$V \in \mathcal{G}_q(n, m)$$

$$V \text{ Sidon space} \iff \mathcal{C} = \mathrm{Orb}(V), \quad [n, 2m-2, \tfrac{q^n-1}{q-1}, m]_q.$$

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This property allowed the authors to

1. Prove the Conjecture **(T.M.B.R.2013)** for most of the cases
2. Provide Explicit Constructions of One-Orbit Cyclic Subspace Codes via Explicit Constructions of Sidon Spaces

Our goals

1. Equivalence of Sidon Spaces via Equivalence of One-Orbit Subspace Codes

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2. *New* Constructions of One-Orbit Cyclic Subspace Codes via *New* Constructions of Sidon Spaces

One-Orbit cyclic subspace codes from Sidon spaces

$S \in \mathcal{G}_q(n, m)$,

S Sidon space $\iff$ $\mathcal{C} = \mathrm{Orb}(S)$, $[n, 2m-2, \frac{q^n-1}{q-1}, m]_q$.

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Equivalence of One-orbit Cyclic Subspace Codes

H. Gluesing-Luerssen and H. Lehmann
Automorphism groups and isometries for cyclic orbit codes
Advances in Mathematics of Communications, 2023.

$U, V \in \mathcal{G}_q(n, m)$

$\mathrm{Orb}(U)$ and $\mathrm{Orb}(V)$ are semilinearly equivalent if

$$\mathrm{Orb}(U) = \mathrm{Orb}(V^\sigma)$$

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Definition

U and V are semilinearly equivalent if there exists

$(\alpha, \sigma) \in \mathbb{F}_{q^n}^* \times \mathrm{Aut}(\mathbb{F}_{q^n})$ such that

$$U = \alpha V^\sigma.$$

Some invariants

$$\mathcal{C} = \mathrm{Orb}(S), \quad S \leqslant_q \mathbb{F}_{q^n} \quad \dim S = m$$

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3. $w(F_S)$ is the weight of F_S, i.e. the number of non-zero coefficients of F_S the monic q-polynomial of q-degree m such that $\ker F_S = S$

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The following integers are invariant under semilinear equivalence:

1. $\dim S^2 \quad \longrightarrow \quad \dim S \geq 3$ and S Sidon space $\Rightarrow 2m \leq \dim S^2 \leq \binom{m+1}{2}$
2. $\delta_k(S) := \dim_{\mathbb{F}_{q^k}} < S >_{\mathbb{F}_{q^k}}$, where $k \mid n$
   S is generic $\Rightarrow 2 \leq \delta_k(S) \leq n/k$
3. $w(F_S)$ is the weight of F_S, i.e. the number of non-zero coefficients of F_S the monic q-polynomial of q-degree m such that $\ker F_S = S \longrightarrow w(F_S) \leq m + 1$

Sidon spaces with $\delta_k(S) = 2$

$V \in \mathcal{G}_q(n, m)$: $\delta_k(V) = 2$ where $k \mid n$ and V is *generic* in $\mathbb{F}_{q^n}$

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Standard form

$$V = V_{U,\gamma} = \{u + v\gamma \,:\, (u, v) \in U\} \subseteq \mathbb{F}_{q^k} + \gamma\mathbb{F}_{q^k}$$

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$$\mathbb{F}_{q^k}(\gamma) = \mathbb{F}_{q^n}, \quad U \leq_{\mathbb{F}_q} \mathbb{F}_{q^k} \times \mathbb{F}_{q^k}, \quad \dim_{\mathbb{F}_q} U = m$$

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$$V = V_{U,\gamma} \text{ Sidon space} \quad \Longrightarrow \quad m = \dim_{\mathbb{F}_q} V = \dim_{\mathbb{F}_q} U \leq k$$

k-dimensional Sidon spaces with $\delta_k(S) = 2$

Theorem

$V \in \mathcal{G}_q(n, k), \quad \delta_k(V) = 2$

$$V = V_{f,\gamma} = \{u + f(u)\gamma \,:\, u \in \mathbb{F}_{q^k}\} \subseteq \mathbb{F}_{q^k}(\gamma)$$

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$$f(x) = \sum_{i=0}^{k-1} a_i x^{q^i} \in \mathcal{L}_{k,q}$$

k-dimensional Sidon spaces with $\delta_k(S) = 2$

Theorem

$$[\mathbb{F}_{q^k}(\gamma) : \mathbb{F}_{q^k}] > 2$$

$$V_{f,\gamma} \text{ is a Sidon space in } \mathbb{F}_{q^k}(\gamma)$$

$$\Updownarrow$$

$$S_{u,f} \cap S_{v,f} = \mathbb{F}_q$$

$$\forall u, v \in \mathbb{F}^*_{q^k} \text{ such that } u\mathbb{F}_q \neq v\mathbb{F}_q$$
$$S_{u,f} = \{\lambda \in \mathbb{F}_{q^k} : f(\lambda u) - \lambda f(u) = 0\}.$$

Sidon polynomials in $\mathbb{F}_{q^k}$

Definition

f q-polynomial in $\mathbb{F}_{q^k}$
f is a Sidon polynomial if

$$S_{u,f} \cap S_{v,f} = \mathbb{F}_q,$$

$\forall u, v \in \mathbb{F}_{q^k}^*$ s.t. $u\mathbb{F}_q \neq v\mathbb{F}_q$
$S_{u,f} = \{\lambda \in \mathbb{F}_{q^k} \colon f(\lambda u) - \lambda f(u) = 0\}$.

Sidon polynomials in $\mathbb{F}_{q^k}$

Definition

f q-polynomial in $\mathbb{F}_{q^k}$
f is a Sidon polynomial if

$$S_{u,f} \cap S_{v,f} = \mathbb{F}_q,$$

$\forall u, v \in \mathbb{F}^*_{q^k}$ s.t. $u\mathbb{F}_q \neq v\mathbb{F}_q$
$S_{u,f} = \{\lambda \in \mathbb{F}_{q^k} : f(\lambda u) - \lambda f(u) = 0\}$.

Corollary

$$V_{f,\gamma} = \{u + f(u)\gamma \ : \ u \in \mathbb{F}_{q^k}\} \subset \mathbb{F}_{q^k}(\gamma)$$

$$[\mathbb{F}_{q^k}(\gamma) : \mathbb{F}_{q^k}] > 2$$

$V_{f,\gamma}$ is a Sidon space in $\mathbb{F}_{q^n}$ iff f is a Sidon polynomial in $\mathbb{F}_{q^k}$.

Sidon spaces with $\delta_k(S) = 2$ and semilinear equivalence

$$U_f = \{(u, f(u)) : u \in \mathbb{F}_{q^k}\} \qquad U_g = \{(u, g(u)) : u \in \mathbb{F}_{q^k}\}$$

Semilinear Equivalence in standard form

$$V_{f,\gamma} = \{u + f(u)\gamma : u \in \mathbb{F}_{q^k}\}$$
$$V_{g,\xi} = \{w + g(w)\xi : w \in \mathbb{F}_{q^k}\}$$

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$V_{f,\gamma}$, $V_{g,\xi}$ semilinearly equivalent if and only if

$\exists A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{F}_{q^k})$ and $\sigma \in \mathrm{Aut}(\mathbb{F}_{q^n})$ such that

$$U_f^\sigma = \{wA : w \in U_g\} = U_g \cdot A$$
$$\xi = \frac{c + d\gamma^\sigma}{a + b\gamma^\sigma}$$

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U_f and U_g are $\Gamma L(2, q^k)$-equivalent

Sidon polynomials

Definition

$f \in \mathcal{L}_{k,q}$ is scattered if for any $a, b \in \mathbb{F}^*_{q^k}$ such that

$$f(a)/a = f(b)/b$$

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$$\not\Leftarrow$$

Known scattered polynomials

	k	$f(x)$	Conditions		$V_{f,\gamma}$
1)		x^{q^s}	$\gcd(s,k)=1$	2000	R.R.T.2017, L.L.2021 Z.T.2023, Z.T.2023
2)		$x^{q^s}+\delta x^{q^{s(k-1)}}$	$\gcd(s,k)=1$, $\mathrm{N}_{q^k/q}(\delta)\neq 1$	2001/2015	
3)	2ℓ	$x^{q^s}+x^{q^{s(\ell-1)}}+$ $\delta^{q^\ell+1}x^{q^{s(\ell+1)}}+\delta^{1-q^{2\ell-1}}x^{q^{s(2\ell-1)}}$	q odd, $\mathrm{N}_{q^{2\ell}/q^\ell}(\delta)=-1$, $\gcd(s,\ell)=1$	2020/2021/2022	
4)	6	$x^q+\delta x^{q^4}$	$q>4$, certain choices of δ	2018-2020	
5)	6	$x^q+x^{q^3}+\delta x^{q^5}$	q odd, $\delta^2+\delta=1$	2018-2020	
6)	8	$x^q+\delta x^{q^5}$	q odd, $\delta^2=-1$	2018	

Sidon spaces from binomials

Theorem

$$f(x) = x^{q^s} + \delta x^{q^l} \in \mathcal{L}_{k,q}, \text{ with } 1 \leq s < l$$

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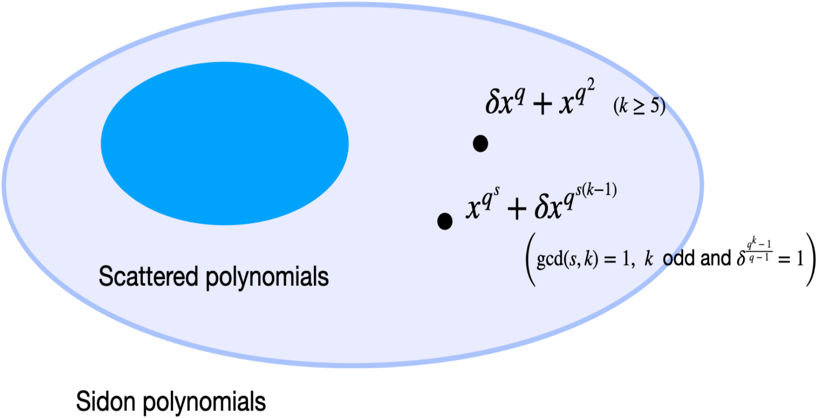
$$f(x) = x^{q^s} + \delta x^{q^l} \in \mathcal{L}_{k,q}, \text{ with } 1 \leq s < l$$

1. $\gcd(k, l-s) = 1 \implies f$ is a Sidon polynomial.
2. $\gcd(k, l-s) = t > 1 \implies f$ is a Sidon polynomial iff f is a scattered polynomial

Sidon spaces from binomials



Comparison with known constructions of Sidon Spaces

$S \in \mathcal{G}_q(n, m),\ m \le k \mid n$ and $\delta_k(S) = 2$;

1. $S \in \mathcal{G}_q(n, m),\ m \le k \mid n$ such that $\delta_k(S) \ge 3$;
2. $S \in \mathcal{G}_q(n, m),\ m \le k \mid n$ as kernel of subspace polynomials with low weight;
3. $S \in \mathcal{G}_q(n, m)$, n possibly prime (mostly existence results)

Thank you for your attention!