A contribution towards the classification of tensors in $\mathbb{F}_q^3 \otimes S^2\mathbb{F}_q^3$, q even

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Basic definitions and notations

Let $V_1, ..., V_t$ be vector spaces over the field $\mathbb{F}_q$; $dim(V_i) = m_i$.

- The *t-order tensor product* $V := V_1 \otimes ... \otimes V_t$ is defined as the set of multilinear functions from $V_1^\vee \times ... \times V_t^\vee$ into $\mathbb{F}_q$, where $V_i^\vee$ is the dual space of V_i.
- *Fundamental (pure or rank-1) tensors* are tensors of the form $v_1 \otimes .... \otimes v_t$.
- The *rank* of a tensor $A \in V$ is the smallest integer r such that

$$A = \sum_{i=1}^{r} A_i \qquad (1)$$

with each A_i a fundamental tensor of V.

QUESTIONS OF INTERESTS:

- Algorithms: given a tensor A, does there exist an algorithm that determines $R(A)$ and decompose it as the sum of fundamental tensors?
- Classifications: can we determine orbits of tensors under some natural group actions:
  - $G :=$ Stabiliser in $\mathrm{GL}(V)$ of the set of rank-1 tensors.

Note:

- $Rank(A) = Rank(\lambda A)$ for $A \in V$ and $\lambda \in \mathbb{F}$.
- Determining the rank of tensors in $V \iff$ Determining the rank of points in $\mathrm{PG}(V)$.
- Example: $\mathrm{PG}(\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3) \cong \mathrm{PG}(17, \mathrm{q})$.

KNOWN CLASSIFICATIONS:

- There are 5 G-orbits of (non-zero) Tensors in $\mathbb{F}_q^2 \otimes \mathbb{F}_q^2 \otimes \mathbb{F}_q^2$ [M. Lavrauw, J. Sheekey, 2014].
- There are 8 G-orbits of (non-zero) Tensors in $\mathbb{F}_q^2 \otimes \mathbb{F}_q^2 \otimes \mathbb{F}_q^3$ [M. Lavrauw, J. Sheekey, 2015].
- There are 17 G-orbits of (non-zero) Tensors in $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$ [M. Lavrauw, J. Sheekey, 2015].

$\mathbb{F}_q^3 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$:

q odd: progress has been made by classifying partially symmetric tensors in $\mathbb{F}_q^3 \otimes S^2\mathbb{F}_q^3$ equivalent to planes of $\mathrm{PG}(5, q)$ containing at least one rank-1 point [M. Lavrauw, T. Popiel, J. Sheekey, 2020].

INTERESTING CONNECTIONS:

$$\text{Tensors} \quad \longleftrightarrow \quad \text{Finite geometric objects}$$

Tensors can represent:

1. subspaces of projective spaces,
2. algebraic varieties,
3. linear systems of hypersurfaces,
4. semifields,
5. arcs.

Tensors and algebraic varieties:

- Fundamental tensors in $\mathrm{PG}(V) \iff$ Points of the Segre variety in $\mathrm{PG}(N,q)$, where $N = \prod dim(V_i) - 1$.
- Example: $\sigma_{1,2,2} : \mathrm{PG}(\mathbb{F}_q^2) \times \mathrm{PG}(\mathbb{F}_q^3) \times \mathrm{PG}(\mathbb{F}_q^3) \longrightarrow \mathrm{PG}(17,q)$
  $$(\langle v_1 \rangle, \langle v_2 \rangle, \langle v_3 \rangle) \mapsto \langle v_1 \otimes v_2 \otimes v_3 \rangle.$$
- Fundamental symmetric tensors in $\mathrm{PG}(V = U \otimes ... \otimes U) \iff$ Points of the Veronese variety in $\mathrm{PG}(M,q)$, where $M = \binom{t+dim(U)-1}{t} - 1$.
- The Veronese surface: $\mathcal{V}(\mathbb{F}_q) \subset S_{2,2}(\mathbb{F}_q)$:
  $$\nu : \mathrm{PG}(2,q) \longrightarrow \mathrm{PG}(5,q)$$
  $$\langle (x_0,x_1,x_2) \rangle \mapsto (x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2).$$
- K:= Stabiliser of $\mathcal{V}(\mathbb{F}_q)$.
- Fundamental alternating tensors in $\mathrm{PG}(V = U \otimes ... \otimes U) \iff$ Points of the Grassmann variety in $\mathrm{PG}(M,q)$, where $M = \binom{dim(U)}{t}$.

TENSORS AND SUBSPACES OF $\mathrm{PG}(5,q)$:

Subspaces of $\mathrm{PG}(5,q)$ are points in $\mathrm{PG}(S^2\mathbb{F}_q^3 \otimes \mathbb{F}_q^r)$.

- $r = 1 \longrightarrow$ points,
- $r = 2 \longrightarrow$ lines,
- $r = 3 \longrightarrow$ planes,
- $r = 4 \longrightarrow$ solids,
- $r = 5 \longrightarrow$ hyperplanes.

Tensors and linear system of conics:

Linear systems of conics := **Subspaces**(PG(2-forms in the projective plane)).

Subspaces of $\mathrm{PG}(5, q)$ correspond to linear systems of conics in $\mathrm{PG}(2, q)$.

- a pencil of conic $\mathcal{P} = \langle C_1, C_2 \rangle$ corresponds to a solid of $\mathrm{PG}(5, q)$.
- a net of conics $\mathcal{N} = \langle C_1, C_2, C_3 \rangle$ corresponds to a plane of $\mathrm{PG}(5, q)$.
- a web of conics $\mathcal{W} = \langle C_1, C_2, C_3, C_4 \rangle$ corresponds to a line of $\mathrm{PG}(5, q)$.

- Classifying linear systems of conics in $\mathrm{PG}(2, q) \iff$ classifying subspaces of $\mathrm{PG}(5, q) \iff$ classifying tensors in $\mathrm{PG}(S^2\mathbb{F}_q^3 \otimes \mathbb{F}_q^r)$.

Previous results on linear systems of conics:

- Dickson (1908): Classified pencils of conics over $\mathbb{F}_q$, q odd.
- Wilson (1914): Incompletely classified rank-one nets of conics (nets with at least a $//$) over $\mathbb{F}_q$, q odd.
- Campbell (1927): Incompletely classified pencils of conics over $\mathbb{F}_q$, q even.
- Campbell (1928): Incompletely classified nets of conics over $\mathbb{F}_q$, q even.

PREVIOUS RESULTS ON ORBITS OF SUBSPACES OF $\mathrm{PG}(5,q)$:

- points, hyperplanes, for all q: ✓
- lines, for all q: ✓ ($\implies$ solids, for q odd: ✓)
  [M. Lavrauw, T. Popiel, 2020]
- planes meeting $\mathcal{V}(\mathbb{F}_q)$ non-trivially, for q odd: ✓
  [M. Lavrauw, T. Popiel, J. Sheekey, 2020]
- solids, for q even: ✓
  [N. Alnajjarine, M. Lavrauw, T. Popiel, 2022]

$\mathrm{PG}(5, odd)$ vs $\mathrm{PG}(5, even)$:

- **q odd:** $\exists$ a polarity: *the set of conic planes of* $\mathcal{V}(\mathbb{F}_q) \to$ *the set of tangent planes of* $\mathcal{V}(\mathbb{F}_q)$.
  - lines $\overset{\text{polarity}}{\Longleftrightarrow}$ solids.
  - $\mathcal{N} = \langle C_1, C_2, C_3 \rangle; C_1 = // \longrightarrow$
    $\pi = H_1 \cap H_2 \cap H_3 \xrightarrow{\text{polarity}}$
    $\pi' = \langle P_1, P_2, P_3 \rangle; P_1 \in \mathcal{V}(\mathbb{F}_q) \longrightarrow$
    Rank-one nets of conics $\Longleftrightarrow$ planes meeting $\mathcal{V}(\mathbb{F}_q)$ non-trivially.
- **q even:** No such polarity $\longrightarrow$
  - lines $\overset{?}{\Longleftrightarrow}$ solids.
  - Rank-one nets of conics $\overset{?}{\Longleftrightarrow}$ planes meeting $\mathcal{V}(\mathbb{F}_q)$ non-trivially.

Representation of Subspaces of $\mathrm{PG}(5,q)$:

- $\mathrm{PG}(5,q) = \langle \mathcal{V}(\mathbb{F}_q) \rangle$.
- Every point $x = (x_0, .., x_5) \in \mathrm{PG}(5,q)$ can be represented by

$$M_x = \begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{bmatrix}$$

- The plane in $\mathrm{PG}(5,q)$ spanned by the 1st three points of the standard frame is

$$\pi = \begin{bmatrix} x & y & z \\ y & . & . \\ z & . & . \end{bmatrix} := \{ \begin{bmatrix} x & y & z \\ y & 0 & 0 \\ z & 0 & 0 \end{bmatrix} : (x,y,z) \in \mathbb{F}_q^3;\ (x,y,z) \in \mathrm{PG}(2,q) \}.$$

Planes of $\mathrm{PG}(5,q)$ and cubic curves in $\mathrm{PG}(2,q)$

$\pi \longrightarrow \mathrm{C} = \mathcal{Z}$(determinant of its matrix representation).

K-ORBITS INVARIANTS:

Let W be a subspace of $\mathrm{PG}(5,q)$, K:=Setwise stabiliser of $\mathcal{V}(\mathbb{F}_q)$ in $\mathrm{PGL}(6,q)$.
Let U_1, U_2,...,U_m denote the distinct K-orbits of r-spaces in $\mathrm{PG}(5,q)$.

- The **rank distribution of** W is

$$[r_1, r_2, r_3]$$

  where

$$r_i = \#\text{ of rank } i \text{ points in } W.$$

- The r**-space orbit-distribution of** W is

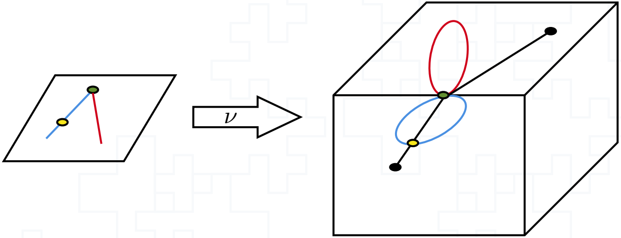
$$[u_1, u_2, \ldots, u_m],$$

  where

  u_i = # of r-spaces incident with W which belong to the orbit U_i.

Properties and Approach:

- **Approach:** We study the possible Point-orbit distributions and discuss the possibility of planes with same Point-OD to split or not under the action of K.
- **Lemma:** Planes with rank distribution $[1, 0, q^2 + q]$ and $[2, r_2 < q, r_3]$ do not exist.
- **Rank-2 points:** The geometry associated with rank-1,2 points can help! ($\pi = \langle Q_1, Q_2, ? \rangle$, where $rank(Q_1) = 1$ and $rank(Q_2) = 2$).

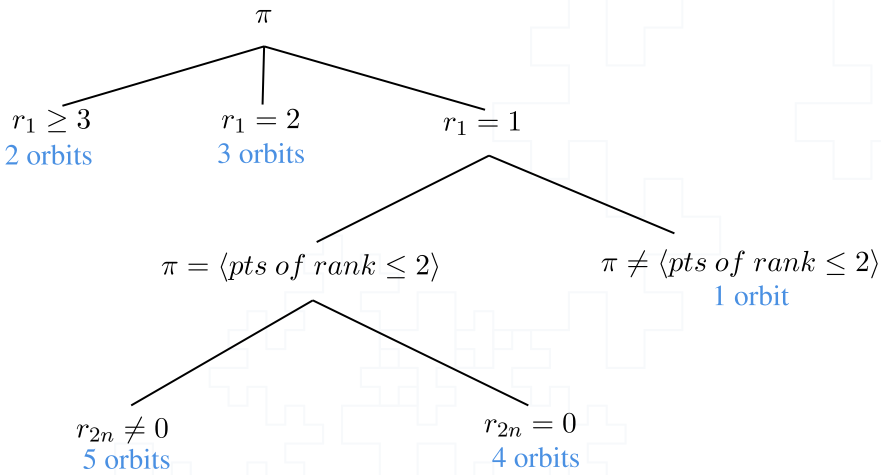


LINES IN $\mathrm{PG}(5,q)$, q EVEN:

Orbits	Point-OD's : $[r_1, r_{2n}, r_{2s}, r_3]$
o_5	$[2, 0, q-1, 0]$
o_6	$[1, 1, q-1, 0]$
$o_{8,1}$	$[1, 0, 1, q-1]$
$o_{8,2}$	$[1, 1, 0, q-1]$
o_9	$[1, 0, 0, q]$
o_{10}	$[0, 0, q+1, 0]$
$o_{12,1}$	$[0, q+1, 0, 0]$
$o_{12,2}$	$[0, 1, q, 0]$
$o_{13,1}$	$[0, 1, 1, q-1]$
$o_{13,2}$	$[0, 0, 2, q-1]$
o_{14}	$[0, 0, 3, q-2]$
o_{15}	$[0, 0, 1, q]$
$o_{16,1}$	$[0, 1, 0, q]$
$o_{16,2}$	$[0, 0, 1, q]$
o_{17}	$[0, 0, 0, q+1]$

Table: K-orbits of lines in $\mathrm{PG}(5,q)$, q even [M. Lavrauw, T. Popiel, 2020].

THE STRUCTURE OF DISCUSSION:



THE CASE $r_{2n} = 0$:

$\pi = \langle Q_1, Q_2, Q_3 \rangle$: $rank(Q_1) = 1$, $rank(Q_i) = 2$, $i = 2, 3$, and $\pi \cap \mathcal{N} = \emptyset$.

- $\mathcal{C}_{Q_2} = \mathcal{C}_{Q_3}$: $Q_1 \in \mathcal{C}_{Q_2}$ or $Q_1 \notin \mathcal{C}_{Q_2} \to \Sigma_6$.
- $Q_1 = U = \mathcal{C}_{Q_2} \cap \mathcal{C}_{Q_3}$.
- $Q_1 \in \mathcal{C}_{Q_2} \setminus \mathcal{C}_{Q_3}$.
- $Q_1 \notin \mathcal{C}_{Q_2} \cup \mathcal{C}_{Q_3}$:
    - $\langle Q_2, Q_3 \rangle \in o_{13,2} : [0, 0, 2, q-1] \iff Q_2 \in T_U(\mathcal{C}_{Q_2})$ and $Q_3 \notin T_U(\mathcal{C}_{Q_3})$, and
    - $\langle Q_2, Q_3 \rangle \in o_{14} : [0, 0, 3, q-2] \iff Q_2 \notin T_U(\mathcal{C}_{Q_2})$ and $Q_3 \notin T_U(\mathcal{C}_{Q_3})$.

THE ORBITS Σ_{12}, Σ_{13} AND Σ_{14}:

- $\pi = \langle Rep\, of\ o_{13,2}, Q_1 \rangle; Q_1 = \nu(a,b,c) \rightarrow \langle Q_1, Q_i \rangle \in o_{8,1} : [1, 0, 1, q-1]$ and thus $a, c \neq 0$

$$\pi_c = \begin{bmatrix} x & y & cx \\ y & y+z & . \\ cx & . & c^2x+z \end{bmatrix}$$

- The cubic curve associated with π_c is:

$$C_c = x(z^2 + yz + c^2y^2) + y^2z.$$

- The Hessian of C_c is:

$$C''_c = x(z^2 + yz + c^2y^2) + z^3 + (1+c^2)y^2z + c^2y^3.$$

- Let $y = 1$ and $\theta = c^{-1}z$: inflexion points of C_c correspond to solutions of $\theta^3 + \theta + c^{-1} = 0$.
- Inflexion points of planes of $\mathrm{PG}(5,q)$ are inflexion points of their associated cubic curves in $\mathrm{PG}(2,q)$.

Cubic equations over $\mathbb{F}_{2^h}$, (Berlekamp, Rumsey, Solomon, 1966)

$$\theta^3 + \theta + c^{-1} = 0,$$

- has three solutions if and only if $q \neq 4$, $Tr(c) = Tr(1)$ and c^{-1} is admissible: $c^{-1} = \frac{v+v^{-1}}{(1+v+v^{-1})^3}$ for some $v \in \mathbb{F}_q \setminus \mathbb{F}_4$,
- a unique solution if and only if $Tr(c) \neq Tr(1)$ and
- no solution if and only if $Tr(c) = Tr(1)$ and c^{-1} is not admissible

Characterization:

- Three inflexions $\rightarrow \Sigma_{14}$; $q \neq 4$.
- A unique inflexion point $\rightarrow \Sigma_{12}$ $(q = 2^{even})$ *or* Σ_{13} $(q = 2^{odd})$.
- No inflexion points $\rightarrow \Sigma_{12}$ $(q = 2^{odd})$ *or* Σ_{13} $(q = 2^{even})$.

THE UNIQUENESS OF Σ_{14}:

$\Sigma_{14} :=$ the union of K-orbits of planes represented by π_c where $h > 2$, $Tr(c) = Tr(1)$ and c^{-1} is admissible.

Proof:

Let L (the inflexion line) be parametrised by $(0, 1, 0)$, $(0, 0, 1)$ and $(0, 1, 1)$ respectively and $Q_{a,b,c} = \nu(a, b, c)$. Then,

$$\pi_{a,b,c} = \langle L, Q_{a,b,c} \rangle \in \Sigma_{14}.$$

If follows that $\langle Q_{a,b,c}, E_i \rangle \in o_{8,1}$; $1 \leq i \leq 3$, and thus $a, b, c \neq 0$.

$$\pi_{b,c} : \begin{bmatrix} x+y & bx & cx \\ bx & b^2x+y+z & bcx \\ cx & bcx & c^2x+z \end{bmatrix}.$$

- $1+b+c=0, \rightarrow \#$.
- $1+b+c \neq 0$, $\mathscr{C}''_{b,c} = \mathcal{Z}(h_{b,c})$, $\alpha = (1+b^2+c^2)$ and $h_{b,c} = c^2\alpha^5xy^2 + \alpha^5xz^2 + c^2(1+b^2)\alpha y^3 + \alpha((1+b^2)+\alpha^3(b^2+c^2))yz^2 + \alpha(c^2(b^2+c^2)+\alpha^3(1+b^2))y^2z + (b^2+c^2)\alpha z^3$.
  Imposing the conditions: $E_i \in \mathscr{C}''_{b,c}$; $1 \leq i \leq 3$, implies that $c^2(1+b^2)\alpha = (b^2+c^2)\alpha = c^2(1+b^2)\alpha + \alpha((1+b^2)+\alpha^3(b^2+c^2)) + \alpha(c^2(b^2+c^2)+\alpha^3(1+b^2)) + (b^2+c^2)\alpha = 0$.
  As $\alpha, c \neq 0$, we get $b=c=1$.

Conclusion:

Φ_{14}: $\Sigma_{14} \longrightarrow o_{14} : \pi \mapsto L$ is a bijection.

UNIQUENESS OF Σ_{12}, Σ_{13}:

$q = 2^{even}$:

- $\pi \in \Sigma_{12}$ has a unique inflexion point $\xrightarrow{\mathbb{F}_{q^2}} \pi(\mathbb{F}_{q^2}) \in \Sigma_{14} \longrightarrow L(\mathbb{F}_{q^2}) \subset \mathrm{PG}(5, q^2)$ is the unique inflexion line in $\pi(\mathbb{F}_{q^2}) \longrightarrow L_s = L(\mathbb{F}_{q^2}) \cap \pi \in \{o_{15}, o_{16,2}\}$. Since $o_{16,2}$ cannot split by extension, $L_s \in o_{15}$.
- Φ_{12}: $\Sigma_{12} \longrightarrow o_{15} : \pi \mapsto L_s$ is a bijection ($o_{15} : [0, 0, 1, q]$).
- Similarly, we can extend our work to $\mathbb{F}_{q^3}$ to conclude Φ_{13}: $\Sigma_{13} \longrightarrow o_{17} : \pi \mapsto L_s$ is a bijection ($o_{17} : [0, 0, 0, q + 1]$).

K-orbits of planes	Representatives	Point-OD	Condition(s)
Σ_1	$\begin{bmatrix} x & y & . \\ y & z & . \\ . & . & . \end{bmatrix}$	$[q+1, 1, q^2-1, 0]$	
Σ_2	$\begin{bmatrix} x & . & . \\ . & y & . \\ . & . & z \end{bmatrix}$	$[3, 0, 3q-3, q^2-2q+1]$	
Σ_3	$\begin{bmatrix} x & . & z \\ . & y & . \\ z & . & . \end{bmatrix}$	$[2, 1, 2q-2, q^2-q]$	
Σ_4	$\begin{bmatrix} x & . & z \\ . & y & z \\ z & z & . \end{bmatrix}$	$[2, 1, 2q-2, q^2-q]$	
Σ_5	$\begin{bmatrix} x & . & z \\ . & y & z \\ z & z & z \end{bmatrix}$	$[2, 0, 2q-2, q^2-q+1]$	
Σ_6	$\begin{bmatrix} x & . & . \\ . & y+cz & z \\ . & z & y \end{bmatrix}$	$[1, 0, q+1, q^2-1]$	$Tr(c^{-1}) = 1$
Σ_7	$\begin{bmatrix} x & y & z \\ y & . & . \\ z & . & . \end{bmatrix}$	$[1, q+1, q^2-1, 0]$	
Σ_8	$\begin{bmatrix} x & y & . \\ y & . & z \\ . & z & . \end{bmatrix}$	$[1, q+1, q-1, q^2-q]$	

Σ_9	$\begin{bmatrix} x & y & . \\ y & z & z \\ . & z & . \end{bmatrix}$	$[1, 1, 2q-1, q^2-q]$	
Σ_{10}	$\begin{bmatrix} x & y & . \\ y & z & . \\ . & . & z \end{bmatrix}$	$[1, 1, 2q-1, q^2-q]$	
Σ_{11}	$\begin{bmatrix} x & y & . \\ y & z & z \\ . & z & x+z \end{bmatrix}$	$[1, 1, q-1, q^2]$	
Σ_{12}	$\begin{bmatrix} x & y & cx \\ y & y+z & . \\ cx & . & c^2x+z \end{bmatrix}$	$[1, 0, q+1, q^2-1]$	$Tr(c) = 1, (*)$
Σ_{13}	$\begin{bmatrix} x & y & cx \\ y & y+z & . \\ cx & . & c^2x+z \end{bmatrix}$	$[1, 0, q-1, q^2+1]$	$Tr(c) = 0, (**)$
Σ_{14}	$\begin{bmatrix} x & y & cx \\ y & y+z & . \\ cx & . & c^2x+z \end{bmatrix}$	$[1, 0, q \mp 1, q^2 \pm 1]$	$Tr(c) = Tr(1), q \neq 4, (***)$
Σ'_{14}	$\begin{bmatrix} x+z & z & z \\ z & y+z & z \\ z & z & y \end{bmatrix}$	$[1, 0, q-1, q^2+1]$	$q = 4$
Σ_{15}	$\begin{bmatrix} x & y & z \\ y & z & . \\ z & . & . \end{bmatrix}$	$[1, 1, q-1, q^2]$	

Comparison with the q odd case:

Rank-one nets of conics $\not\Leftrightarrow$ planes meeting $\mathcal{V}(\mathbb{F}_q)$ non-trivially.

$\pi_6 \in \Sigma_6$ meets $\mathcal{V}(\mathbb{F}_q)$ in a unique point, however its associated net of conics $\mathcal{N}_6$ defined by

$$\alpha X_0 X_1 + \beta X_0 X_2 + \gamma(X_1^2 + cX_1X_2 + X_2^2) = 0$$

has $q+1$ pairs of real lines defined by the pencil

$$\mathcal{Z}(X_0X_1, X_0X_2)\ (\in \Omega_4),$$

and a unique pair of conjugate imaginary lines given by

$$\mathcal{Z}(X_1^2 + cX_1X_2 + X_2^2),$$

implying that $\mathcal{N}_6$ is not a rank-1 net of conics.

Planes meeting $\mathcal{V}(\mathbb{F}_q)$ non-trivially $\Longleftrightarrow$ Nets of conics in $\mathrm{PG}(2,q)$ with a non-empty base.

TO SUM UP:

1. There is an interesting interplay between tensors and geometric objects.
2. There are 15 K-orbits of planes having at least one rank-1 point in $\mathrm{PG}(5,q)$ and 5 when $q=2$.
3. Unlike the q odd case, rank-one nets of conics $\not\Leftrightarrow$ planes meeting $\mathcal{V}(\mathbb{F}_q)$ non-trivially, q even.
4. Planes meeting $\mathcal{V}(\mathbb{F}_q)$ non-trivially $\Longleftrightarrow$ Nets of conics in $\mathrm{PG}(2,q)$ with a non-empty base.
5. Planes of type Σ_{14} (resp. $\{\Sigma_{12},\Sigma_{13}\}$) $\Longleftrightarrow$ Lines of type o_{14} (resp. $\{o_{15},o_{17}\}$).
6. Remaining part of the classification: planes disjoint from $\mathcal{V}(\mathbb{F}_q)$, for all q.

Thank you!

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