Finite Hilbert's incidence geometry & friends

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FINITE GEOMETRY & FRIENDS
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- Axioms ($\mathscr{A}$):
  - I_1: For every two points A, B there exists a line a that contains each of the points A, B.
  - I_2: For every two points A, B there exists no more than one line that contains each of the points A, B.
  - I_3: There exist at least two points on a line. There exist at least three points that do not lie on a line.
  - I_4: For any three points A, B, C that do not lie on the same line there exists a plane α that contains each of the points A, B, C. For every plane there exists a point which it contains.
  - I_5: For any three points A, B, C that do not lie on one and the same line there exists no more than one plane that contains each of the three points A, B, C.
  - I_6: If two points A, B of a line a lie in a plane α then every point of a lies in the plane α.
  - I_7: If two planes α, β have a point A in common then they have at least one more point B in common.
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- We are interested in finite models (P, L, Pl) of $\mathscr{A}$.

The 4-point model

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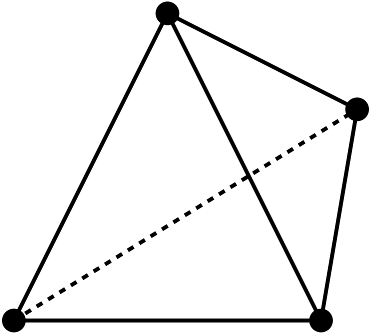
The smallest finite model of $\mathscr{A}$:



Tetrahedron-models

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Theorem

Let n be an integer, $n \geqslant 4$. Let i be an integer, $2 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$. Let:

$$P = \{1, 2, \dots, n\},$$
$$L = \{\{1, 2, \dots, i\}, \{i+1, i+2, \dots, n\}\} \cup \{\{x, y\} : 1 \leqslant x \leqslant i,\ i+1 \leqslant y \leqslant n\},$$
$$\mathrm{Pl} = \{\{1, 2, \dots, i, x\} : i+1 \leqslant x \leqslant n\} \cup \{\{i+1, i+2, \dots, n, y\} : 1 \leqslant y \leqslant i\}.$$

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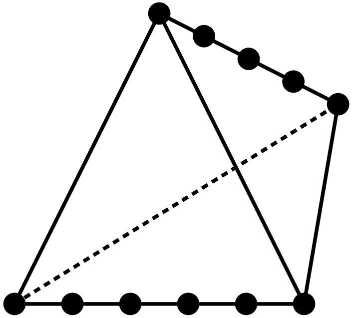

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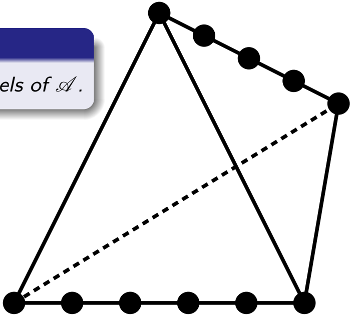
Proposition

There are $\lfloor \frac{n-2}{2} \rfloor$ nonisomorphic tetrahedron-models of $\mathscr{A}$.



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Theorem

Let F^4 be a 4-dimensional vector space over some finite field F of order q. Let P be the set of 1-dimensional subspaces of F^4, let L be the set of 2-dimensional subspaces, and let Pl be the set of 3-dimensional subspaces. Then (P, L, Pl) is a model of $\mathscr{A}$.

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Proposition

Up to isomorphism, there is one n-element projective-space-model of $\mathscr{A}$ for each number n of the form q^3+q^2+q+1, where q is a prime power.

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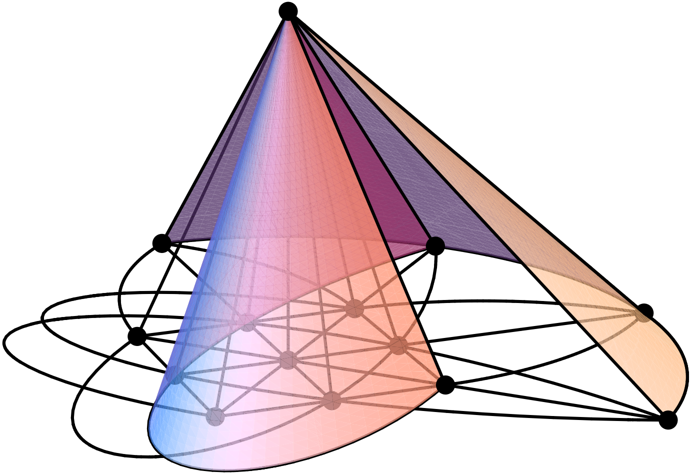
Proposition

For each n of the form $q^2 + q + 2$, where q is a number such that there exists a projective plane of order q, there are as many n-element projective-plane-models of $\mathscr{A}$ as there are nonisomorphic projective planes with $n - 1$ points.

Extensions of projective planes

The projective-plane-model with 14 points:



Combinatorial designs

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- The pair $D = (X, \beta)$, with $|X| = v$ and $\beta \subseteq P_{=k}(X)$, is called a t-(v, k, λ) *design*, and the members of β are called *blocks*, if every t-subset of X occurs in exactly λ blocks. We assume $v > k > t \geqslant 1$ and $\lambda \geqslant 1$.

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Let (X, β) *be a quasi-symmetric* 3-$(v, k, 1)$ *design with intersection numbers* 0 *and* 2. *Let* $P = X$, *let* L *be the set of all two-element subsets of* X *and let* $\mathrm{PI} = \beta$. *Then* (P, L, PI) *is a model of* $\mathscr{A}$.

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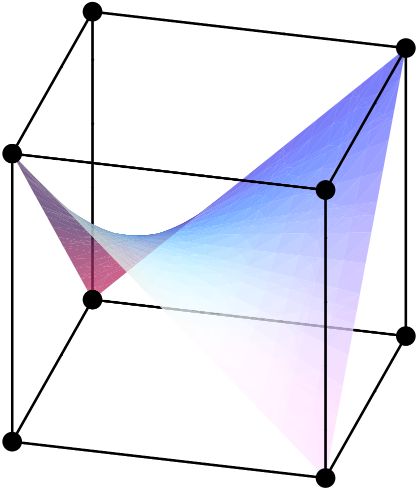
Proposition

There are exactly two nonisomorphic design-models of $\mathscr{A}$. These are the 3-$(8, 4, 1)$ design and the 3-$(22, 6, 1)$ design, corresponding to $n = 8$ and $n = 22$, respectively.

Combinatorial designs

The design-model with 8 points:



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Theorem

Let n be a positive integer. Then:

$$\mathrm{HilbInc}(n) \geqslant \left\lfloor \frac{n-2}{2} \right\rfloor + i + j + k,$$

where

$$i = \begin{cases} 1, & \text{if } n = q^3 + q^2 + q + 1 \text{ for some prime power } q; \\ 0, & \text{otherwise;} \end{cases}$$

$$j = \begin{cases} \text{the number of projective planes of order } q, & \text{if } n = q^2 + q + 2 \text{ for some } q \text{ for which exists a projective plane of order } q; \\ 0, & \text{otherwise;} \end{cases}$$

$$k = \begin{cases} 1, & \text{if } n = 8 \text{ or } n = 22; \\ 0, & \text{otherwise.} \end{cases}$$

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$$r(X) = \max\{|Y| : Y \subseteq X,\ Y \in \mathscr{I}\}.$$

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- Closure operator: $\mathrm{cl} : P(E) \mapsto P(E)$,

$$\mathrm{cl}(X) = \{x \in E : r(X \cup \{x\}) = r(X)\}.$$

Closed set: $\mathrm{cl}(X) = X$.

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a) *Let* Mod, $\text{Mod} = (P, L, \text{PI})$, *be a model of* $\mathscr{A} \setminus \{I_7\} \cup \{I_{\therefore}\}$. *Let* $M_{\text{Mod}} = (P, \mathscr{I})$, *where*

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b) *Let* M, $M = (E, \mathscr{I})$, *be a simple matroid of rank* 4. *Let* $\mathrm{Mod}_M = (E, L, \mathrm{Pl})$, *where*

$$L = \{X : X \text{ is a closed subset of } E \text{ of rank } 2\}$$

and

$$\mathrm{Pl} = \{X : X \text{ is a closed subset of } E \text{ of rank } 3\}.$$

Then Mod_M *is a model of the axiom set* $\mathscr{A} \setminus \{I_7\} \cup \{I_{\therefore}\}$.

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b) *Let* M, $M = (E, \mathscr{I})$, *be a simple matroid of rank* 4. *Let* $\mathrm{Mod}_M = (E, L, \mathrm{Pl})$, *where*

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- Each model of $\mathscr{A}$ is also a model of $\mathscr{A} \cup \{I_{\therefore}\}$.
- There are 185,981 simple matroids of rank 4 with 9 elements (and a negligible number of them with less elements), and thus the same number of models of $\mathscr{A} \setminus \{I_7\} \cup \{I_{\therefore}\}$. We select those which additionally satisfy I_7, and thus obtain the number of models of $\mathscr{A}$ with up to 9 elements.

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- Running the algorithm on all the 28,872,972 simple matroids of rank 3 with 12 elements took about ten days on 16 cores (and the time spent on matroids with less elements was insignificant).

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$$P = \{1, 2, 3, \dots, 12\};$$

$$L = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\},\{5,6\},\{5,7\},\{5,8\},\{6,7\},\{6,8\},\{7,8\},\{9,10\},\{9,11\},\{9,12\},\{10,11\},\{10,12\},\{11,12\},\{1,5,9\},\{1,6,12\},\{1,7,10\},\{1,8,11\},\{2,5,11\},\{2,6,10\},\{2,7,12\},\{2,8,9\},\{3,5,12\},\{3,6,9\},\{3,7,11\},\{3,8,10\},\{4,5,10\},\{4,6,11\},\{4,7,9\},\{4,8,12\}\};$$

$$\mathrm{Pl} = \{\{1,2,3,4\},\{5,6,7,8\},\{9,10,11,12\},\{1,2,5,8,9,11\},\{1,2,6,7,10,12\},\{1,3,5,6,9,12\},\{1,3,7,8,10,11\},\{1,4,5,7,9,10\},\{1,4,6,8,11,12\},\{2,3,5,7,11,12\},\{2,3,6,8,9,10\},\{2,4,5,6,10,11\},\{2,4,7,8,9,12\},\{3,4,5,8,10,12\},\{3,4,6,7,9,11\}\}.$$

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The unexpected 12-element model: